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Approximation of Intuitionistic Fuzzy Bézier Curve Model

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ABSTRACT

In this paper, approximation of Bézier curve by using intuitionistic fuzzy approach is introduced. Firstly, intuitionistic fuzzy point relation is defined based on intuitionistic fuzzy concept to yield intuitionistic fuzzy control point. Some theorems of intuitionistic fuzzy point relation are also introduced. Later, the intuitionistic fuzzy control point is blended with Bernstein blending function and intuitionistic fuzzy Bézier curve model is produced. Next, the approximation curves consists of membership, non-membership and uncertainty curves is visualized. A numerical example is shown and the introduced Bézier curve properties are also discussed. Finally, an algorithm to obtain intuitionistic fuzzy Bézier curve is presented at the end of this paper.

Keywords: Approximation; Bézier curve; intuitionistic fuzzy Bézier curve; intuitionistic fuzzy control point.

1. Introduction

Curve and its properties play a major role in visualizations of data such as from medical, marines, geological, design, manufacturing, physical and other natural behavior or phenomenon. Approximation is an important term used in definition of curves. A curve is compute by using points where minimum points required is $n = 3$. The points used are also called control points and the curve generated passes approximate to the points and not through them (Salomon (2006)). The curve defined by the control point later is called approximating curve. One of the most reliable curve modeling techniques that often used is Bézier curve.

The geometric properties of Bézier were developed by Casteljau (1959) and Casteljau (1963), and later by Pierre Bézier starting in 1962 as stated in Hoschek and Lasser (1993). Casteljau developed a system mainly aim in the design of curves and surfaces. He also used Bernstein polynomials in order to define curve and surface and introduced de Casteljau algorithm. The Bézier method use control polygons where the control polygon utilizes points near it rather than defining a curve through points on it (Farin (2002)). By using control polygon in Bézier method, the curve shape can be changed by changing the control polygon. This property have made the Bézier model very useful in real application.

Pierre Bézier has derived the mathematical basis of curves and surfaces techniques from geometrical considerations as in Bezier (1968), Bezier (1970) and Bezier (1971). Later, around 1970s, Forrest (1972) and Gordon and Reisenfeld (1974) found the connection between the work of Bézier and the classical Bernstein polynomials. They discovered that the Bernstein polynomials are in fact the basis functions used for Bézier curves and surfaces. Curve is a necessary and inevitable in order to represent data point (Hoschek and Lasser (1993)). However, the nature of data point obtained is difficult to understand, process and represent as it is affected by noise and uncertainty. Normally, data with uncertainty characteristic will be ignored or removed from a set of data disregarding its effect on the resulting curve and surface. Hence, the evaluation and analyzing process will be incomplete. If there exist an element of uncertainty, the data set should be filtered so that it can be used to generate curve of a model that want to be investigated. Therefore an appropriate approach is needed to visualize and overcome this problem.

Intuitionistic fuzzy set (IFS) is a generalization of fuzzy set theory from Zadeh (1965) and was introduced by Atanassov (1983). IFS considers membership function, non-membership and non-determinacy function in analyzing information while fuzzy set theory only considers the membership function (Atanassov (1986), Atanassov (1999), Atanassov (2012)). Many research and work related to IFSs have been carried out such as from Atanassov and Gargov (1989), Szmidt and Kacrzyk (2000), Atanassov (2000), Cornelis et al. (2004), Yuan et al. (2014), Bashir et al. (2015), Diaz et al. (2015), Rahman (2016), Ngan (2016), Wang and Chen (2017) and Hassaballah and Ghareeb (2017). IFS is defined by membership function, non-membership function and uncertainty function with the constraint that summation of these three functions must equal to one (Ciftcibasi and Altunay (1998)).

Research of IFS with geometric modeling have been done by Zulkifly and Wahab. Zulkifly and Wahab (2015) introduced an idea of IFS in spline curve and surface which focus on Bézier spline where the curve and surface are blended with intuitionistic fuzzy control point (IFCP). Wahab et al. (2016) discussed intuitionistic fuzzy Bézier model and generated intuitionistic fuzzy Bézier curve (IFBC) using interpolation method. They visualized IFBC by blending the Bernstein polynomial IFCP that have been defined. Later, Zulkifly and Wahab (2018) defined IFCP through intuitionistic fuzzy concept with some properties and illustrated intuitionistic fuzzy bicubic Bézier surface by using approximation method. Using IFCP, Wahab and Zulkifly (2018a) and Wahab and Zulkifly (2018b) also generated cubic Bézier curve through interpolation method and intuitionistic fuzzy B-spline curve using approximation method.

Curve modeling is a method of mathematical representations construction in the form of geometry while IFS theory is a mathematical representation that aimed at concepts and techniques to tackles uncertain problems. The aim of this paper is to generate and illustrate IFBC through approximation method by using IFCP. This paper is organized as follows. Section 1 discussed some introduction and previous works related to this research. In section 2, intuitionistic fuzzy point relation (IFPR), its properties and IFCP is shown. Section 3 introduces approximation of IFBC by using IFCP. Section 4 shows a numerical example and visualization of IFBC. The properties of the curve and the algorithm to obtain the curve are also shown. Finally, section 5 will conclude this research.

2. Intuitionistic Fuzzy Point Relation

IFPR is developed based on the concept of IFS and defined as follows:

Definition 2.1. *Let V, W be a collection of points with non-empty set and*

$V, W, I \subseteq \mathbb{R}^3$, then IFPR is defined as

$$T^* = \Big\{ \big\langle (v_i, w_j), \mu_T(v_i, w_j), \pi_T(v_i, w_j) \big\rangle | \\ \big((\mu_T(v_i, w_j), \nu_T(v_i, w_j), \pi_T(v_i, w_j) \big) \in I \Big\} \tag{1}$$

where (v_i, w_j) are points in ordered pair and $(v_i, w_j) \in V \times W$. $\mu_T(v_i, w_j), \nu_T(v_i, w_j)$ and $\pi_T(v_i, w_j)$ are the grades of membership, non-membership and uncertainty of the ordered pair of points respectively in $[0, 1] \in I$. The condition $0 \leq \pi_T + \nu_T(v_i, w_j) \leq 1$ is follow and the degree of uncertainty is denoted by

$$\pi_T(v_i, w_j) = 1 - \big(\mu_T(v_i, w_j) + \nu_T(v_i, w_j)\big) \tag{2}$$

IFPR is based on fuzzy point in the Euclidean space and intuitionistic fuzzy point (IFP) is in IFS. Hence, IFPR is in intuitionistic fuzzy relation (IFR) and represented by $T^* \in R^*$ and $P^* \times Q^* \in A^* \times B^*$.

Definition 2.2. *Let P^* be an IFP and A^* is intuitionistic fuzzy number (IFN) in V. P^* is said to be in A^* and denoted by $P^* \in A^*$ if and only if $\mu_P(v_i) \leq \mu_A(v_i)$ and $\nu P(v_i) \geq \nu_A(v_i)$ for all $v_i \in V$. Every A^* can be expressed as the union of all IFP that belong to A^* which if $\mu_A(v_i)$ and $\nu_A(v_i)$ is non-zero for $v_i \in V$, then $\mu_A(v_i) = sup\{y : \mu_P(v_i)$ is IFP (membership) and $0 < y \leq \mu_A(v_i)\}$ and $\nu_A(v_i) = inf\{y : \nu_P(v_i)$ is IFP (non-membership) and $0 < \mu_A(v_i) \leq y\}$ respectively. Therefore, each and every IFP P^* in A^* can be written as $P^* = \{P_i^* | i = 1, 2, ..., n, i \in \mathbb{N}\}$ and $A^* = P_1^* \cup P_2^* \cup ... \cup P_n^*$.*

Theorem 2.1. *If $A^* = \bigcup_{i \in I} A_i^*$ where $I = \{1, 2, ..., n\}$ and I is any index, then $P^* \in A^*$ if and only if $P^* \in A_i^*$ for some $i \in I$.*
Proof: *Let the support for P^* denoted by v_0, then*

$$\mu_A(v_0) = \underset{i \in I}{sup}(\mu_{A_i}(v_0)), \\ \nu_A(v_0) = \underset{i \in I}{inf}(\nu_{A_i}(v_0)) \tag{3}$$

(i) There exists some $i_0 \in I$ such as $\mu_{A_{i_0}}(v_0) = \mu_A(v_0)$ and $\nu_{A_{i_0}}(v_0) = \nu_A(v_0)$. (ii) $\mu_{A_i}(v_0) \leq \mu_{A_i}(v_0)$ and $\nu_{A_i}(v_0) \geq \nu_{A_i}(v_0)$ for all $i \in I$. For (i). $P^* \in A_{i_0}^*$. For (ii), $P^* \in A^*$ implies that $\mu_P(v_0) \leq \mu_A(v_0)$, $\nu_P(v_0) \geq \nu_A(v_0)$ and considering that $\mu_A(v_0) = \underset{i \in I}{sup} \mu_{A_i}(v_0), \nu_A(v_0) = \underset{i \in I}{inf} \nu_{A_i}(v_0)$, it follows that $\mu_P(v_0) \leq \mu_{A_{i_0}}(v_0), \nu_P(v_0) \geq \nu_{A_{i_0}}(v_0)$ for some i_0. Thus, $P^* \in A_{i_0}^*$

Definition 2.3. *Let P^* and Q^* be an IFP and A^* and B^* is IFN in V and W respectively. Hence, IFPR T^* on P^* and $Q^*, P^* \times Q^*$ is said to be in R^*, and denoted by $P^* \times Q^* \in A^* \times B^*$ if and only if $\mu_T(v_i, w_j) \leq$*

$\mu_R(v_i, w_j)$ and $\nu_T(v_i, w_j) \geq \nu_R(v_i, w_j)$ for all $(v_i, w_j) \in V \times W$. Obviously, every R^* can be expressed as the union of all IFPR that belong to R^* which if $\mu_T(v_i, w_j)$ and $\nu_T(v_i, w_j)$ is non-zero for $(v_i, w_j) \in V \times W$, then $\mu_R(v_i, w_j) = sup\{\mu_{P\times Q}(v_i, w_j) : \mu_{P\times Q}(v_i, w_j)$ is IFPR (membership) and $0 < \mu_{P\times Q}(v_i, w_j) \leq \mu_R(v_i, w_j)$ and $\nu_R(v_i, w_j) = inf\{\nu_{P\times Q}(v_i, w_j) : \nu_{P\times Q}(v_i, w_j)$ is IFPR (non-membership) and $0 < \mu_R(v_i, w_j) \geq \nu_{P\times Q}(v_i, w_j)$ respectively. Therefore, each and every T^* in R^* can be written as $T^* = \{T_i^* | i = 1, 2, ..., n, i \in \mathbb{N}\}$ and $R^* = T_1^* \cup T_2^* \cup ... \cup T_n^*$.

Theorem 2.2. *If $R^* = \bigcup_{i\in I} A_i^*$ where $I = \{1, 2, ..., n\}$ and I is any index, then $T^* \in R^*$ if and only if $T^* \in R_i^*$ for some $i \in I$*
Proof: *Let the support for T^* denoted by (v_0, w_0), then*

$$\begin{aligned} \mu_R(v_0, w_0) &= \sup_{i\in I}(\mu_{R_i}(v_0, w_0)), \\ \nu_R(v_0, w_0) &= \inf_{i\in I}(\nu_{R_i}(v_0, w_0)) \end{aligned} \tag{4}$$

(i) There exists some $i_0 \in I$ such as $\mu_{R_{i_0}}(v_0, w_0) = \mu_R(v_0, w_0)$ and $\nu_{R_{i_0}}(v_0, w_0) = \nu_R(v_0, w_0)$. (ii) $\mu_{R_i}(v_0, w_0) \leq \mu_{R_i}(v_0, w_0)$ and $\nu_{R_i}(v_0, w_0) \geq \nu_{R_i}(v_0, w_0)$ for all $i \in I$. For (i). $T^* \in R_{i_0}^*$. For (ii), $T^* \in R^*$ implies that $\mu_T(v_0, w_0) \leq \mu_R(v_0, w_0)$, $\nu_T(v_0, w_0) \geq \nu_R(v_0, w_0)$ and considering that $\mu_R(v_0, w_0) = \sup_{i\in I} \mu_{R_i}(v_0, w_0), \nu_R(v_0, w_0) = \inf_{i\in I} \nu_{R_i}(v_0, w_0)$, it follows that $\mu_T(v_0, w_0) \leq \mu_{R_{i_0}}(v_0, w_0), \nu_T(v_0, w_0) \geq \nu_{R_{i_0}}(v_0)$ for some i_0. Thus, $T^* \in R_{i_0}^*$.

The collection of all point or set of points that are used to determine the shape of a curve is called control point. The control point plays an important role in the process of generating, controlling and producing smooth curve. IFCP is defined as follows:

Definition 2.4. *Let T^* be an IFPR, then IFCP is defined as set of points $n+1$ that indicates the positions and coordinates of a location and is used to describe the curve and is denoted by*

$$C_i^* = \{C_0^*, C_1^*, ..., C_n^*\} \tag{5}$$

where the control polygon vertices or the control point is numbered from 0 to n.

3. Intuitionistic Fuzzy Bézier Curve Model

A Bézier curve is a parametric curve with polynomial function of parameter t that was used in geometric modelling and is determined by its control polygon

(Yamaguchi (1998),Rogers (2001)). The degree of the polynomial depends on the number of points to define the curve. IFBC is obtained by blending IFCP with the Bernstein polynomial or basis function and shown in the next definition.

Definition 3.1. *Let $C^* = \left\{C_i^*\right\}_i^*$ where $i = 0, 1, ..., n$ be the IFCP, and IFBC denoted by $B^*(t)$ with the position vector along the curve as a function of the parameter t, hence by blended C^* with the blending function, IFBC is written as*

$$B^*(t) = \sum_{i=0}^{n} C_i^* B_i^n(t), 0 \leq t \leq 1 \tag{6}$$

where
i) $B_i^n = \begin{pmatrix} n \\ i \end{pmatrix} t^i(1-t)^{n-i} \equiv 1$ is the blending function with $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!(n-i)!}, 0 \equiv 1$ is the binomial coefficient.
ii) C^* is i^{th} IFCP called geometric coefficient that forms control polygon for IFBC degree n.
iii) IFBC with n^{th} degree is written as $B^*(t) = T_0^* B_0^n + T_1^* B_1^n + ... + T_n^* B_i^n$.

The IFBC in (6) consists of membership, non-membership and uncertainty surface and denoted as follows:

$$B^{\mu}(t) = \sum_{i=0}^{n} C_i^* B_i^n(t) \tag{7}$$

$$B^{\nu}(t) = \sum_{i=0}^{n} C_i^* B_i^n(t) \tag{8}$$

$$B^{\pi}(t) = \sum_{i=0}^{n} C_i^* B_i^n(t) \tag{9}$$

Theorem 3.1. *For each Bernstein basis function IFBC, $B_i^n > 0$ for $t \in (0, 1)$ for all $n \leq 0$ and $i = 0, 1, ..., n$.*
Proof: *For $t \in (0, 1)$,$t > 0$ and $(1 - t) > 0$. Therefore B_i^n is results of n positive factor product and it is positive.*

Theorem 3.2. *$\sum_{i=0}^{n} B_i^n(t)$ for $t \in (0, 1)$.*
Proof: *The proof is follows from binomial teorem where $(r + s)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} r^{n-1}s^i$. Let $r = (1 - t)$ and $s = t$, therefore, $((i - t) + t)^n = 1^n = 1 = (r + s)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (1 - t)^{n-i}t^i = \sum_{i=0}^{n} B_i^n(t)$.*

Intuitionistic fuzzy control polygon for IFBC is a polygon that is obtained by connecting the IFCP in order. The degree of the polynomial defining IFBC segment is one less than the number of intuitionistic fuzzy control polygon points. It can be said that IFBC generally follows the shape of intuitionistic fuzzy control polygon. The first and the last IFCP are coincident with the first and last points of intuitionistic fuzzy control polygon. Moreover, the IFBC is contained within the convex hull of the intuitionistic fuzzy control polygon.

Definition 3.2. *Let C^* be an IFCP, hence the piecewise curve that is shaped by line segments through C_i^* and C_{i+1}^* for $i = 0, 1, ..., n-1$ is called intuitionistic fuzzy control polygon.*

Theorem 3.3. *All planar of IFBC lies in convex hull of IFCP $\left\{C_i^*\right\}_i^*$ associated with intuitionistic fuzzy control polygon.*
Proof: *Each and every point that lies on IFBC is computed by using de Casteljau algorithm. de Casteljau algorithm that is based on line construction connecting points at the intuitionistic fuzzy control polygon vertices. Obviously, at every step computed, these lines must lies in the convex hull of IFCP. Therefore Theorem 3.3 is proven.*

Fig. 1 visualize an example of intuitionistic fuzzy control polygon consists of membership, non-membership and uncertainty control polygon denoted by blue, green and red dots respectively. These control polygon will control the generated IFBC.

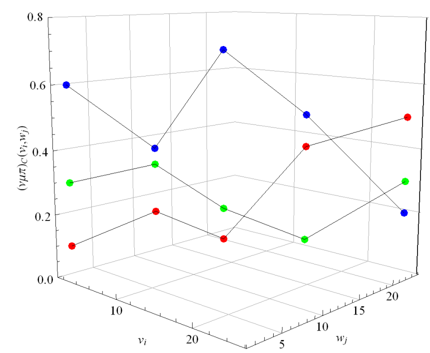

Figure 1: Intuitionistic fuzzy control polygons

Theorem 3.3 is visualized in Fig. 2 until Fig. 5 for $t = 0.3, t = 0.5, t = 0.7$ and $t = [0, 1]$ for intuitionistic fuzzy Bézier polygon to generate membership curve. Specifically, this method use linear interpolation. de Casteljau algorithm is a dynamical operation to calculate point on the IFBC and recursive algorithms to construct point on IFBC.

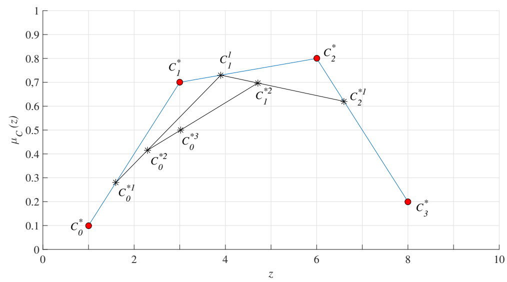

Figure 2: de Casteljau algoritm to generate membership Bézier curve for $t = 0.3$

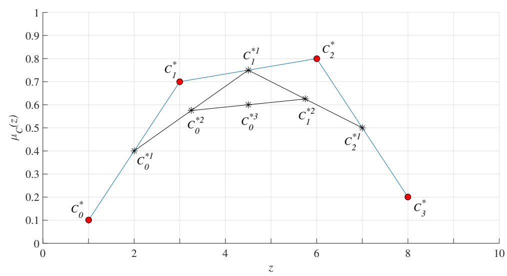

Figure 3: de Casteljau algoritm to generate membership Bézier curve for $t = 0.5$

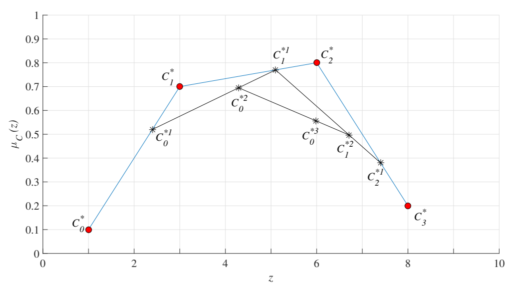

Figure 4: de Casteljau algoritm to generate membership Bézier curve for $t = 0.7$

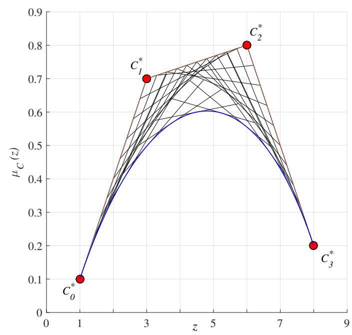

Figure 5: de Casteljau algoritm to generate membership Bézier curve for $t = [0, 1]$

4. Numerical Example, Properties and Algorithm of Intuitionistic Fuzzy Bézier Model

In order to illustrate IFBC approximation, let's consider IFBC with five IFCP and degree four ($n = 4$) as in Table I.

Table 1: Intuitionistic Fuzzy Control Point with its Respective Degrees

IFCP	Membership	Non-Membership	Uncertainty
$C_0^* = (2, 2)$	(0.6)	(0.3)	(0.1)
$C_0^* = (7, 8)$	(0.4)	(0.4)	(0.2)
$C_0^* = (11, 13)$	(0.7)	(0.2)	(0.1))
$C_0^* = (17, 18)$	(0.5)	(0.1)	(0.4)
$C_0^* = (25, 23)$	(0.2)	(0.3)	(0.5)

From Table 1 and through (6), the desired IFBC's approximation curve is visualized separately from Fig. 6 until Fig. 8 with their respective control points and intuitionistic control polygon by blending the IFCP with the Bernstein blending function. Fig. 6 until Fig. 8 represent the membership, non-membership and uncertainty Bézier curve. From those figures, the shape of the curve generated by the intuitionistic fuzzy control polygon can be predict easily.

Next, the IFBC is visualized as in Fig. 9. Fig 9. shows IFBC with its respective IFCP and intuitionistic fuzzy control polygon.

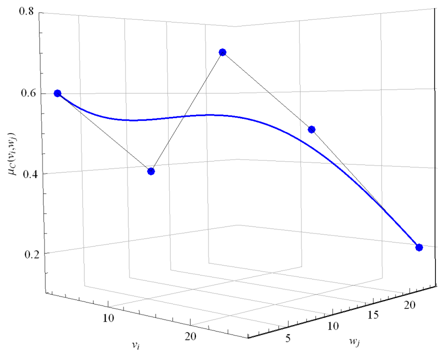

Figure 6: Membership Bézier curve with its respective control points and control polygon

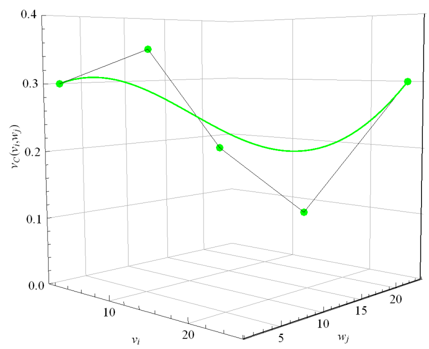

Figure 7: Non-membership Bézier curve with its respective control points and control polygon

IFBC is determined by its intuitionistic fuzzy control polygon. The properties of IFBC's consists of three curve (membership, non-membership and uncertainty) while the Bézier curve is only for the crisp curve. Because the Bézier basis is also the Bernstein basis, some of the IFBC are known and summarized as follows:

i) The first and the last points on IFBC are coincident with the first and last point of the intuitionistic fuzzy control polygon.

ii) The IFBC generally follows the shape of the intuitionistic fuzzy control polygon.

iii) IFBC lies in the convex hull for all parametric value .

iv) The degree of the polynomial defining the curve segment is one less than the number of intuitionistic fuzzy control polygon points.

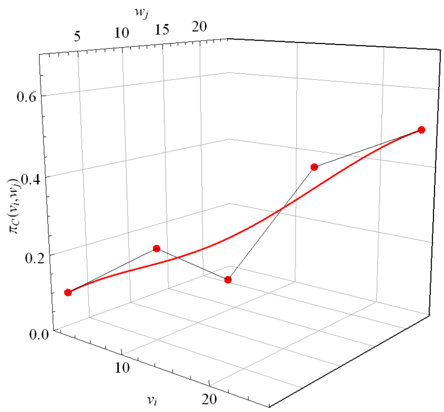

Figure 8: Uncertainty Bézier curve with its respective control points and control polygon.

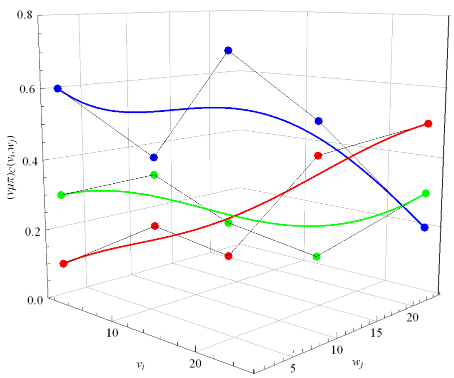

Figure 9: Intuitionistic fuzzy Bézier curve with its respective intuitionistic fuzzy control points and intuitionistic fuzzy control polygon

v) The IFBC is invariant under an affine transformation.
vi) The Bernstein basis function in non-negative and its sum is equal to one.
vii) The generated IFBC exhibits the variation-diminishing property.
viii) The tangent vector at the ends of IFBC have the same direction as the first and the last intuitionistic fuzzy control polygon spans respectively.

An algorithm to obtain IFBC is simplified in matrix form as below:

1. IFCP or the vertices of intuitionistic fuzzy control polygon is determined

with
$$C^* = \left\{C_i^*\right\}_i^*$$

2. Step 2. The equation of IFCP is expressed in matrix forms as
$$B^*(t) = [K][L][C^*] = [M][C^*]$$
where C^* is IFCP and M is the Bernstein basis function $B_i^n(t)$.

3. Step 3. Later, the coefficient of the parameter terms are collected and rewrite as
$$B^*(t) = [K][L][C^*]$$
where K is the parameter and L is the coefficient.

4. Step 4. Step 1 until Step 3 is repeated for membership, non-membership and uncertainty curve.

5. Conclusion

This paper has introduced approximation of IFBC model by defining IFCP. Approximation of IFBC model is an ideal approach in modeling data involving intuitionistic features because it is characterized by membership function, non-membership function and uncertainty function. Through these functions, all data information provided will be processed and analyzed compared to crisp Bézier curve. Approximation of IFBC model can be applied in stochastic processes, real time tracking, stock market, remote sensing, data mining, databases, management decision-making field, economy, routing and wireless sensor networks. The intuitionistic fuzzy data combine with visualization using Bézier can gives complete information in analyzing and describing the nature of problems studied with its reasoning.

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